Syntactic Forcing Models for Coherent Logic

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Categories of forcing conditions

- Fix a finite first-order signature Σ
- Fix a countably infinite set of variables $\mathsf{X} = \{x_0, x_1, \dots\}$
- Let $\mathrm{Tm}(X)$ be the set of Σ-terms over $X \subseteq \mathsf{X}$
- Define the category $\mathbb{C}_{\mathrm{ts}}$ having:
  - Objects denoted as pairs $(X; A)$, where X is a finite subset of X and A is a finite set of atoms in the language defined by Σ and X. (This means that only variables from X may occur in A.) Such pairs $(X; A)$ are called *conditions*.
  - Morphisms denoted as $f : (Y; B) \to (X; A)$, where f is a *term substitution* $X \to \mathrm{Tm}(Y)$ such that $Af \subseteq B$
  - Composition $f \circ g$ of $g : (Z; C) \to (Y; B)$ with f above is the substitution $X \to \mathrm{Tm}(Z)$ that is the composition fg (in diagram order!) of the respective substitutions
  - Indentity morphisms $(X; A) \to (X; A)$ are identity substitutions $X \to X$
- Similarly to $\mathbb{C}_{\mathrm{ts}}$, define $\mathbb{C}_{\mathrm{vs}}$ ($\mathbb{C}_{\mathrm{rn}}$) when in addition $f(X) \subseteq Y$ (and also f injective)

Categories of forcing conditions (ctnd)

- $\mathbb{C}_{\mathrm{ts}}, \mathbb{C}_{\mathrm{vs}}, \mathbb{C}_{\mathrm{rn}}$ are indeed categories with terminal object $(;)$
- Conditions will be denoted as, e.g., $(x, y, z; p(z), q(f(x), z, z))$
- Substitutions will be denoted as, e.g., $[y := x, z := g(x)]$
- Post-fixing substitutions in diagram order: $(Af)g = A(fg)$
- Depending on Σ, categories $\mathbb{C}_{\mathrm{ts}}, \mathbb{C}_{\mathrm{vs}}, \mathbb{C}_{\mathrm{rn}}$ are rather different:
  - $[x := 0], [x := 1] : (;) \rightrightarrows (x;)$ cannot be equalized
  - $[x := y], [x := z] : (y, z;) \rightrightarrows (x;)$ can be equalized by $[y := w, z := w] : (w;) \to (y, z;)$ in $\mathbb{C}_{\mathrm{vs}}$, but not in $\mathbb{C}_{\mathrm{rn}}$
- Actually, $\mathbb{C}_{\mathrm{ts}}$ $(\mathbb{C}_{\mathrm{vs}})$ has all finite products (limits)
- Depending on Σ, categories $\mathbb{C}_{\mathrm{ts}}, \mathbb{C}_{\mathrm{vs}}, \mathbb{C}_{\mathrm{rn}}$ will lead to different forcing semantics (good for independence proofs!)
- What does a condition mean? A finite, partial description of potential models. Time to consider a coherent theory T ...

Coverages depending on coherent theories

- Fix a coherent theory T
- Define inductively a relation $\lhd_T$ between conditions and finite sets of conditions (denoted by $U, V, ...$):

(base) $C \lhd_T \{C\}$ for all conditions C

(step) If T has an axiom $\forall \vec{x}.\ (C \to (\exists \vec{y}_1.B_1) \vee \cdots \vee (\exists \vec{y}_n.B_n))$ such that for some sequence of terms $\vec{t}$ with variables in X we have $C[\vec{t}/\vec{x}] \subseteq A$, then the following rule applies:

$$\frac{(X, \vec{y}_1; A, B_i[\vec{t}/\vec{x}]) \lhd_T U_1 \quad \ldots \quad (X, \vec{y}_n; A, B_i[\vec{t}/\vec{x}]) \lhd_T U_n}{(X; A) \lhd_T \bigcup_{1 \leq i \leq n} U_i}$$

- Looks familiar? Let's take the semantic point of view.
- Example: if $T = \{p \to (q \vee r)\}$, then $(;p) \lhd_T \{(;p,q),(;p,r)\}$. The models of T extending $(;p)$ are models extending $(;p,q)$ or models extending $(;p,r)$
- Borderline case: if $T = \{p \to \bot\}$, then $(;p) \lhd_T \emptyset$
- When $C \lhd U$ (drop $_T$, also: $U \rhd C$) we say that U covers C

Structural properties of the coverage

- The properties of $\lhd$ use (any one of) $\mathbb{C}_{\mathrm{ts}}, \mathbb{C}_{\mathrm{vs}}, \mathbb{C}_{\mathrm{rn}}$
- Lemma $\lhd 1$. If $(X;A) \lhd U$ and $(Y;B) \in U$, then $X \subseteq Y$ and $A \subseteq B$ and $i_{X,Y} : (Y;B) \to (X;A)$. Prf: easy induction on $\lhd$.
- Lemma $\lhd 2$. If $f : D \to C$ and $C \lhd U$, then there is $V \rhd D$ such that, for any $E \in V$ there is an $F \in U$ such that $g : E \to F$ with g an extension of f. Proof: induction on $\lhd$. Intuition: view D as an (extension of) the f-instance of C. NB $\mathbb{C}_{\mathrm{rn}}$ OK!
- Lemma $\lhd 3$. If $C \lhd U$ and for every $D \in U$ we have a $V_D \rhd D$, then $C \lhd \bigcup_{D \in U} V_D$. Proof: induction on $\lhd$. Intuition: transitivity.
- Together with $\lhd 0 : \ C \lhd \{C\}$, $\lhd$0–$\lhd$3 provide what is needed for the coming definition of forcing to give a sound and complete semantics.
- Further abstraction $\rightsquigarrow$ Grothendieck topology and site

Forcing relation based on coverage

Let $\lhd$ be a coverage. For any condition $C = (X; A)$ and any first-order formula ϕ with free variables in X, we define the forcing relation $C \Vdash \phi$ by induction on ϕ as follows:

1. $C \Vdash \top$
2. $C \Vdash \bot$ if $C \lhd \emptyset$ (i.e., $A \vdash_X \bot$, explain)
3. $C \Vdash \phi$ if ϕ is an atom and there is $U \rhd C$ such that $\phi \in B$ for all $(Y; B) \in U$ (i.e., $A \vdash_X \phi$)
4. $C \Vdash \phi_1 \wedge \phi_2$ if $C \Vdash \phi_1$ and $C \Vdash \phi_2$
5. $C \Vdash \phi_1 \vee \phi_2$ if for some U we have $C \lhd U$ and ($D \Vdash \phi_1$ or $D \Vdash \phi_2$) for all $D \in U$
6. $C \Vdash \phi_1 \to \phi_2$ if for all D and morphisms $f : D \to C$ we have $D \Vdash \phi_2 f$ whenever $D \Vdash \phi_1 f$
7. $C \Vdash \forall x.\phi$ if for all $D = (Y; B)$ and morphisms $f : D \to C$ we have $D \Vdash \phi[f, x = t]$ for all $t \in \mathrm{Tm}(Y)$
8. $C \Vdash \exists x.\phi$ if there is $U \rhd C$ such that, for all $D \in U$, $D = (Y; B)$, $D \Vdash \phi[x = t]$ for some $t \in \mathrm{Tm}(Y)$

Examples

- The law of the excluded middle is not forced: for $\Sigma = \{p\},\ T = \emptyset$, not $(;) \Vdash p \vee \neg p$
- Unlike Kripke semantics, there is no one-world frame. Hence for $\Sigma = \{p\},\ T = \emptyset$, surprisingly, $(;) \Vdash \neg\neg p$
- Classical contingencies can sometimes be forced: for $\Sigma = \{P(\text{-})\},\ T = \emptyset$, never $C \Vdash \forall x.P(x)$, so $(;) \Vdash (\forall x.P(x)) \to \bot$
- Distinguishing $\Vdash_{\mathrm{vs}}$ and $\Vdash_{\mathrm{ts}}$: if $\Sigma = \{Z(\text{-}), 0\}$ and
  - $T = \{\neg Z(0)\}$, then $(x;) \Vdash_{\mathrm{vs}} \neg\neg Z(x)$, and not $(x;) \Vdash_{\mathrm{ts}} \neg\neg Z(x)$ (since $[x := 0] : (;) \to (x;)$, $(;) \Vdash_{\mathrm{ts}} \neg Z(0)$ and not $(;) \Vdash_{\mathrm{ts}} \bot$)
  - Better, add $\exists x.\top$ to T and get for $\phi = \exists x.\neg\neg Z(x)$ that $(;) \Vdash_{\mathrm{vs}} \phi$ and not $(;) \Vdash_{\mathrm{ts}} \phi$.
- NB $T \vdash \exists x.\top$ and yet it makes a difference
- Distinguishing $\Vdash_{\mathrm{rn}}$ from $\Vdash_{\mathrm{vs}}, \Vdash_{\mathrm{ts}}$ is done in [BBC, 6.4] by a rather complicated example (with relational Σ)

Special soundness: forcing the theory itself

- Fix a coherent theory T with its $\lhd$ and $\Vdash$
- For all $\phi \in T$ we have $(;) \Vdash \phi$
- Proof by example: take $\phi \equiv \forall x.\ (P(x) \to (p \vee \exists y.Q(x,y)))$. (TL;DR) Note that $(;)$ is final, so we have to show that $C \Vdash P(t) \to p \vee \exists y.Q(t,y)$ for all conditions $C = (X;A)$ and $t \in \mathrm{Tm}(X)$. So, we have to show that $D \Vdash (p \vee \exists y.Q(tf,y))$ for all $D = (Y;B)$ and $f : D \to C$ with $D \Vdash P(tf)$. Now, if $U \rhd D$ such that every $E \in U$ contains $P(tf)$, then we can use the instance of ϕ with tf to cover E such that $E \Vdash p \vee \exists y.Q(tf,y)$, and use $\lhd 3$ to get $D \Vdash p \vee \exists y.Q(tf,y)$.
- By the general soundness result (next slide), not only T is forced, but also all its intuitionistic, possibly non-coherent consequences.

General soundness of the forcing semantics

- Fix a signature Σ, one of the categories $\mathbb{C}_{\text{ts}}, \mathbb{C}_{\text{vs}}, \mathbb{C}_{\text{rn}}$, a coverage $\lhd$ with its forcing relation by $\Vdash$
- No coherent theory T is assumed here
- Let $\Gamma \vdash^i_X \phi$ denote intuitionistic provability (explain X)
- Soundness: for all formulas Γ, ϕ with free variables in X, if $\Gamma \vdash^i_X \phi$, then for any C and $\rho : X \to \mathrm{Tm}(C)$,

$$C \Vdash \Gamma\rho \text{ implies } C \Vdash \phi\rho$$

- Proof: induction on $\Gamma \vdash^i_X \phi$ (long and tedious)

Completeness for coherent formulas

- Fix a coherent theory T with its $\lhd$ and $\Vdash$
- Coherent completeness: for every coherent sentence ϕ, if $(,) \Vdash \phi$, then $T \vdash^i_\emptyset \phi$
- For the proof we need a version for open formulas
- Completeness: for every coherent sentence ϕ with free variables in X, any condition $C = (Y; A)$ and $\rho : X \to \mathrm{Tm}(Y)$,

$$C \Vdash \phi\rho \text{ implies } T, A \vdash^i_X \phi\rho$$

- Proof by induction on ϕ
- This proof is constructive, and doesn't use 'fairness'
- On the other hand, there is syntax in this semantics

Redundant sentences

- Let T be a coherent theory. A sentence ϕ is called T-*redundant* if all coherent sentences ψ such that $T, \phi \vdash^i \psi$ can be proved already in T
- The combination of soundness with coherent completeness yields that ϕ is redundant if ϕ is forced: if $T \vdash^i \phi \to \psi$, then by soundness $\phi \to \psi$ is forced. Hence if ϕ is forced, then also ψ is forced, and hence provable in T if coherent.
- Example (Kock): in the theory of local rings, the following formula is forced and hence redundant (suprise?)

$$\neg(x = 0 \wedge y = 0) \to (\exists z.xz = 1) \vee (\exists z.yz = 1)$$